J. Differential Equations 252 (2012) 4503–4528

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Quaternionic-valued ordinary differential equations II. Coinciding sectors

Paweł Wilczyński

Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Krakow, Poland

ARTICLE INFO

Article history:
Received 27 September 2011
Available online 27 January 2012

Keywords:
Periodic solutions
Quaternions
Polynomial equations

ABSTRACT

We give some sufficient conditions for the existence of at least one periodic solution of the quaternionic polynomial equations. In some cases we are able to prove uniqueness of periodic solutions inside some sectors of $\mathbb{H}$ and detect solutions heteroclinic to the periodic ones.

1. Introduction

Presented paper is a continuation of [1,2]. We investigate the existence of periodic solutions of the quaternionic equation

$$\dot{q} = v(t, q) \tag{1}$$

where v is a polynomial of order $n \geqslant 3$ in q and is periodic in t. We write it in the form

$$\dot{q} = v(t, q) = \sum_{l=0}^{n} \sum_{\mu=0}^{l} q^{\mu} a_{l,\mu}(t) q^{l-\mu} \tag{2}$$

where coefficients $a_{l,\mu}$ are quaternionic-valued and continuous functions.

The study of Eq. (1) was initiated by Campos and Mawhin [3]. Then it occurred that the case of Riccati equation (i.e. $n = 2$) appears in the Euler vorticity dynamics (see [4]). The complete description of dynamics of the Riccati equation was given in [5]. Papers [6] and [7] are devoted to the autonomous version of (1).

E-mail address: pawel.wilczynski@yahoo.pl.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2012.01.005

We present partial extensions to the quaternionic settings of the main results of [2]. We use topological tools such as Ważewski method and Brouwer fixed point theorem. The latter one is applied to the Poincaré map in some special subset of $\mathbb{H}$. In the complex case if the vector field is holomorphic, then the Poincaré map is also holomorphic, thus the Brouwer fixed point theorem can be strengthen to the Wolff–Denjoy one and the uniqueness of periodic solution inside some regions of the phase space can be obtained (cf. [8]). But the quaternionic case seems much more complicated. While the concept of construction of special subsets in the complex case can be carried over into the quaternionic case there is no quaternionic version of the Wolff–Denjoy theorem. There are some theories of "regular" quaternionic-valued functions imitating the theory of holomorphic functions (see e.g. [9–11]) but either the Poincaré map is not regular in their sense or the theory does not contain the quaternionic version of the Wolff–Denjoy theorem. That is why we obtain mainly only existence of periodic solutions inside some sets. But in some cases we are able to prove the uniqueness of periodic solutions inside them. It is possible due to the fact that the Poincaré map is contracting in some sets (cf. Lemma 9). This idea comes from the geometric properties of the polynomial vector field and was developed in the complex case and carried over into the quaternionic one.

The paper is organised as follows. In Section 2 we collect some basics facts concerning Ważewski method, processes and quaternions. We also state a crucial fixed point lemma which is used instead of the Wolff–Denjoy theorem. In Section 3 we present main results concerning existence of periodic solutions. Firstly, we deal with the case of nonzero free coefficient of the vector field and obtain existence of periodic solutions inside some cones. Secondly, we investigate the case of zero free term. In Section 4 we present some improvement of the previous approach, namely, by imposing some additional restrictions on the width of cones we are able to prove also uniqueness of periodic solutions inside them.

2. Basic facts

2.1. Dynamical systems and Ważewski method

Let X be a topological space and W be a subset of X. We denote by $\operatorname{cl} W$ the closure of W. The following definitions come from [12].

Let D be an open subset of $\mathbb{R} \times X$. By a *local flow* on X we mean a continuous map $\phi : D \to X$, such that three conditions are satisfied:

(i) $I_x = \{t \in \mathbb{R}\colon\ (t, x) \in D\}$ is an open interval (α_x, ω_x) containing 0, for every $x \in X$,
(ii) $\phi(0, x) = x$, for every $x \in X$,
(iii) $\phi(s + t, x) = \phi(t, \phi(s, x))$, for every $x \in X$ and $s, t \in \mathbb{R}$ such that $s \in I_x$ and $t \in I_{\phi(s,x)}$.

In the sequel we write $\phi_t(x)$ instead of $\phi(t, x)$.

Let ϕ be a local flow on X, $x \in X$ and $W \subset X$. We call the set

$$\phi^+(x) = \phi\big([0, \omega_x) \times \{x\}\big)$$

the *positive semitrajectory* of $x \in X$.

We distinguish three subsets of W given by

$$W^- = \big\{x \in W\colon \phi\big([0, t] \times \{x\}\big) \not\subset W, \text{ for every } t > 0\big\},$$
$$W^+ = \big\{x \in W\colon \phi\big([-t, 0] \times \{x\}\big) \not\subset W, \text{ for every } t > 0\big\},$$
$$W^* = \big\{x \in W\colon \phi(t, x) \notin W, \text{ for some } t > 0\big\}.$$

It is easy to see that $W^- \subset W^*$. We call W^- the *exit set of* W, and W^+ the *entrance set of* W.

We call W a *Ważewski set* provided

1. if $x \in W$, $t > 0$, and $\phi([0,t] \times \{x\}) \subset \operatorname{cl} W$ then $\phi([0,t] \times \{x\}) \subset W$,
2. W^- is closed relative to W^*.

Proposition 1. *If both W and W^- are closed subsets of X then W is a Ważewski set.*

The function $\sigma : W^* \to [0,\infty)$

$$\sigma(x) = \sup\bigl\{t \in [0,\infty)\colon\ \phi\bigl([0,t] \times \{x\}\bigr) \subset W\bigr\}$$

is called the *escape-time function of W*.

The following lemma is called the Ważewski lemma.

Lemma 2. *(See [12, Lemma 2.1(iii)].) Let W be a Ważewski set and σ be its escape-time function. Then σ is continuous.*

2.2. Processes

Let X be a topological space and $\Omega \subset \mathbb{R} \times X \times \mathbb{R}$ be an open set.

By a *local process* on X we mean a continuous map $\varphi : \Omega \to X$, such that three conditions are satisfied:

(i) $I_{(\sigma,x)} = \{t \in \mathbb{R}\colon\ (\sigma, x, t) \in \Omega\}$ is an open interval containing 0, for every $\sigma \in \mathbb{R}$ and $x \in X$,
(ii) $\varphi(\sigma, \cdot, 0) = \mathrm{id}_X$, for every $\sigma \in \mathbb{R}$,
(iii) $\varphi(\sigma, x, s+t) = \varphi(\sigma + s, \varphi(\sigma, x, s), t)$, for every $x \in X$, $\sigma \in \mathbb{R}$ and $s, t \in \mathbb{R}$ such that $s \in I_{(\sigma,x)}$ and $t \in I_{(\sigma+s,\varphi(\sigma,x,s))}$.

For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma, x, t)$.

Local process φ on X generates a local flow ϕ on $\mathbb{R} \times X$ by the formula

$$\phi\bigl(t, (\sigma, x)\bigr) = \bigl(\sigma + t, \varphi(\sigma, x, t)\bigr).$$

Let M be a smooth manifold and let $v : \mathbb{R} \times M \to TM$ be a time-dependent vector field. We assume that v is so regular that for every $(t_0, x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = v(t, x), \tag{3}$$

$$x(t_0) = x_0 \tag{4}$$

has unique solution. Then Eq. (3) generates a local process φ on X by $\varphi_{(t_0,t)}(x_0) = x(t_0, x_0, t + t_0)$, where $x(t_0, x_0, \cdot)$ is the solution of the Cauchy problem (3), (4).

Let T be a positive number. In the sequel T denotes the period. We assume that v is T-periodic in t. It follows that the local process φ is T-periodic, i.e.,

$$\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)} \quad \text{for all } \sigma, t \in \mathbb{R},$$

hence there is a one-to-one correspondence between T-periodic solutions of (3) and fixed points of the Poincaré map $\varphi_{(0,T)}$.

2.3. Quaternions

We follow [3] and use the letters q, p to denote (real) quaternions. By

$$q = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4 \tag{5}$$

we mean a quaternion

$$q = q_0 + q_1 i + q_2 j + q_3 k, \tag{6}$$

where the symbols i, j, k satisfy the following rules of multiplication

$$i^2 = j^2 = k^2 = ijk = -1,$$
$$ij = -ji = k, \qquad ki = -ik = j, \qquad jk = -kj = i.$$

We denote the set of quaternions by $\mathbb{H}$.

For a quaternion q we define the *scalar part*

$$\mathfrak{s}_q = q_0 \in \mathbb{R}$$

and *vectorial part* by

$$\mathfrak{v}_q = (q_1, q_2, q_3) \in \mathbb{R}^3.$$

Thus one can write

$$q = (\mathfrak{s}_q, \mathfrak{v}_q) \in \mathbb{R} \times \mathbb{R}^3 \tag{7}$$

and the multiplication of two quaternions q, p has the form

$$qp = (\mathfrak{s}_q, \mathfrak{v}_q)(\mathfrak{s}_p, \mathfrak{v}_p) = (\mathfrak{s}_q\mathfrak{s}_p - \mathfrak{v}_q \cdot \mathfrak{v}_p, \mathfrak{s}_q\mathfrak{v}_p + \mathfrak{s}_p\mathfrak{v}_q + \mathfrak{v}_q \times \mathfrak{v}_p),$$

where $\cdot$ and $\times$ denote the inner product and cross product in $\mathbb{R}^3$, respectively.

In the sequel we use all the notations (5), (6) and (7) so one can think of $\mathbb{H}$ as $\mathbb{R}^4$ or $\mathbb{R} \times \mathbb{R}^3$.

We introduce the *inner product* and *modulus* of the quaternions q, $p = p_0 + p_1 i + p_2 j + p_3 k$ by

$$\langle q, p\rangle = q_0 p_0 + q_1 p_1 + q_2 p_2 + q_3 p_3 = \mathfrak{s}_q\mathfrak{s}_p + \mathfrak{v}_q \cdot \mathfrak{v}_p,$$
$$|q| = \sqrt{\langle q, q\rangle} = \sqrt{\mathfrak{s}_q^2 + |\mathfrak{v}_q|^2},$$

where $|\ |$ denotes also the norm in $\mathbb{R}^n$ (and of course a standard modulus in $\mathbb{C}$). Thus $\mathbb{H}$ is a Hilbert space.

For a quaternion q we introduce the *real part* operator

$$\mathfrak{Re}(q) = q_0$$

and *imaginary part* by

$$\mathfrak{Im}(q) = q_1 i + q_2 j + q_3 k.$$

Thus $q = \mathfrak{Re}(q) + \mathfrak{Im}(q)$ and its *conjugate* has the form

$$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k = (\mathfrak{s}_q, -\mathfrak{v}_q) = \mathfrak{Re}(q) - \mathfrak{Im}(q).$$

Moreover,

$$\left[\mathfrak{Im}(q)\right]^2 = -\left|\mathfrak{Im}(q)\right|^2 \tag{8}$$

holds. In general, the multiplication of two quaternions q, p is not commutative but its projection on the scalar part has this property, i.e.

$$\mathfrak{Re}(qp) = \mathfrak{Re}(pq).$$

However, it is not true in the case of three or more quaternions e.g. $\mathfrak{Re}[ijk] = -1 \neq 1 = \mathfrak{Re}[jik]$ but for $r \in \mathbb{H}$ we can write

$$\mathfrak{Re}[pqr] = \mathfrak{Re}[rpq] = \mathfrak{Re}[qrp], \tag{9}$$

i.e. only cyclic permutations are allowed. Moreover, quaternion q commutes with all other quaternions if and only if $q \in \mathbb{R}$.

Now we list some useful formulae

$$\langle q, p \rangle = \mathfrak{Re}(q\bar{p}) = \mathfrak{Re}(\bar{q}p),$$
$$\overline{(qp)} = \bar{p}\bar{q}, \qquad q\bar{q} = \bar{q}q = |q|^2, \qquad |qp| = |pq| = |p||q|,$$
$$\frac{1}{q} = \frac{\bar{q}}{|q|^2}.$$

$\mathbb{H}$ is a noncommutative field and it is evident that it contains $\mathbb{R}$ as a subfield. It also contains $\mathbb{C}$ as a subfield.

For a nonzero quaternion q we introduce the *argument* by

$$\mathrm{Ark}(q) = \left|\mathrm{Arg}\left(\mathfrak{s}_q + |\mathfrak{v}_q| i\right)\right|.$$

It is easy to see that $\mathrm{Ark}(q) \in [0, \pi]$ and $\mathrm{Ark}(0)$ is not defined. The straightforward calculation shows that the following proposition holds.

Proposition 3. *Let $p, q \in \mathbb{H}$ be such that $\mathrm{Ark}(p) + \mathrm{Ark}(q) \leqslant \frac{\pi}{2}$. Then*

$$\mathrm{Ark}(pq) = \mathrm{Ark}(qp) \leqslant \mathrm{Ark}(p) + \mathrm{Ark}(q) \tag{10}$$

holds.

Both the equality and strict inequality may happen in (10) as shown in the following example.

Example 4. Let us notice that $\mathrm{Ark}(1+i) = \mathrm{Ark}(1+j) = \frac{\pi}{4}$ holds. Then

$$\mathrm{Ark}\left((1+i)^2\right) = \frac{\pi}{2}$$

and

$$\operatorname{Ark}\bigl((1+i)(1+j)\bigr) = \operatorname{Ark}(1+i+j+k) = \frac{\pi}{3}$$

are satisfied.

For $0 < \alpha \leqslant \pi$ we define the sector $\mathcal{S}(\alpha)$ by

$$\mathcal{S}(\alpha) = \bigl\{q \in \mathbb{H}\colon \operatorname{Ark}(q) < \alpha\bigr\}.$$

Obviously, $0 \notin \mathcal{S}(\alpha)$.

For $q \in \mathbb{H}$ we define the *exponential* of q by

$$e^q = \exp(q) = \sum_{n=0}^{\infty} \frac{q^n}{n!},$$

where the series converges absolutely and uniformly on compact subsets of $\mathbb{H}$. If $p \in \mathbb{H}$ and $pq = qp$ then $e^p e^q = e^q e^p = e^{p+q}$. Let $\alpha \in \mathbb{R}$ and $I \in \mathbb{H}$ be such that $I^2 = -1$. Then $e^{\alpha I} = \cos(\alpha) + \sin(\alpha) I$.

Let $q, p \in \mathcal{C}^1(\mathbb{R}, \mathbb{H})$ then $qp \in \mathcal{C}^1(\mathbb{R}, \mathbb{H})$ and

$$(qp)'(t) = q(t)p'(t) + q'(t)p(t). \tag{11}$$

2.4. Quaternionic power

Let α, β be quaternions such that $|\alpha| = |\beta| = 1$. Then the map $g : \mathbb{H} \ni q \to \alpha q \beta \in \mathbb{H}$ is an orthogonal rotation. Moreover, every orthogonal rotation in $\mathbb{H}$ has this form (cf. [13, Chapter 10]). When $\alpha = \overline{\beta}$ then the rotation $\alpha q \beta$ affects only the vectorial part of q. The straightforward calculation shows that the following proposition holds.

Proposition 5. *Let $\alpha, q \in \mathbb{H} \setminus \{0\}$. Then*

$$\operatorname{Ark}(\alpha q \overline{\alpha}) = \operatorname{Ark}(q) \tag{12}$$

holds.

Now let us observe that $I^2 = -1$ if and only if

$$I \in \bigl\{q \in \mathbb{H}\colon |q| = 1,\ \mathfrak{Re}[q] = 0\bigr\}.$$

Thus $\mathbb{R}[I] \subset \mathbb{H}$ and $\mathbb{C}$ are isomorphic.

Remark 6. Quaternionic multiplication in $R[I]$ is commutative, i.e. for every $x, y \in R[I]$ the equality $xy = yx$ holds.

For a given $q \in \mathbb{H} \setminus \mathbb{R}$ we write $I_q = \frac{\mathfrak{Im}\, q}{|\mathfrak{Im}\, q|}$. We define an isomorphism g_q between $\mathbb{R}[I_q]$ and $\mathbb{C}$ by

$$g_q : \mathbb{R}[I_q] \ni p = x + I_q y \mapsto x + iy \in \mathbb{C},$$

where $x, y \in \mathbb{R}$.

It is easy to see that for every $p \in \mathbb{R}[I_q]$ the equality

$$\operatorname{Ark}[p] = \left|\operatorname{Arg}\left[g_q(p)\right]\right| \tag{13}$$

holds.

Now we show that g_q may be defined in terms of rotation. Namely, let $\lambda_q \in \mathbb{H}$ be such that

$$|\lambda_q| = 1 \tag{14}$$

and

$$\lambda_q I_q \overline{\lambda_q} = i \tag{15}$$

hold (to see that such λ_q exists see e.g. [13, Chapter 10]). The number λ_q may be not uniquely determined, e.g. for every number of the form $\alpha j + \beta k$ satisfying $\alpha^2 + \beta^2 = 1$, $\alpha, \beta \in \mathbb{R}$ the formula

$$(\alpha j + \beta k)(-i)\overline{(\alpha j + \beta k)} = i$$

holds, so, since $I_{1-i} = -i$, every such number may be chosen as λ_{1-i}. Nevertheless, we can write

$$g_q(p) = \lambda_q p \overline{\lambda_q},$$

so

$$g_q^{-1}(p) = \overline{\lambda_q} p \lambda_q.$$

It has sense on the whole $\mathbb{H}$ but we use it only on $\mathbb{R}[I_q]$. Different choices of λ_q give the same value of $g_q(p)$ provided that $p \in \mathbb{R}[I_q]$ and (14), (15) are satisfied. To see this let us calculate for $q \in \mathbb{H} \setminus \mathbb{R}$, and $p = x + I_q y \in \mathbb{R}[I_q]$, $x, y \in \mathbb{R}$

$$\lambda_q p \overline{\lambda_q} = \lambda_q (x + I_q y)\overline{\lambda_q} = \lambda x \overline{\lambda_q} + y \lambda I_q \overline{\lambda_q} = x + iy = g_q(x + I_q y) = g_q(p).$$

Now let us recall that for $m \in \mathbb{R}$ and $z \in \mathbb{C}$ we write $z^m = e^{m \operatorname{Log}(z)}$ wherever it has sense.

Let $m \in \mathbb{R}$ and $q \in \mathbb{H}$. We define

$$\Lambda(q, m) = \begin{cases} g_q^{-1}([g_q(q)]^m), & \text{for } q \in \mathbb{H} \setminus \mathbb{R}, \\ q^m, & \text{for } q \in \mathbb{R}. \end{cases}$$

Now for $q \in \mathbb{H} \setminus \mathbb{R}$ the equalities

$$\begin{aligned} \Lambda(q, 2) &= g_q^{-1}\left(\left(\mathfrak{Re}(q) + i\left|\mathfrak{Im}(q)\right|\right)^2\right) = g_q^{-1}\left(\mathfrak{Re}^2(q) - \left|i(q)\right|^2 + 2i\,\mathfrak{Re}(q)\left|\mathfrak{Im}(q)\right|\right) \\ &= \mathfrak{Re}^2(q) - \left|i(q)\right|^2 + 2\,\mathfrak{Re}(q)\left|\mathfrak{Im}(q)\right| g_q^{-1}(i) = \mathfrak{Re}\left(q^2\right) + 2\,\mathfrak{Re}(q)\left|\mathfrak{Im}(q)\right| I_q \\ &= q^2 \end{aligned}$$

and

$$\Lambda(1 - i, 6) = q_{1-i}^{-1}\left([1 + i]^6\right) = q_{1-i}^{-1}(-8i) = -8q_{1-i}^{-1}(i) = 8i = (1 - i)^6$$

hold. This illustrates that Λ is a generalisation of quaternionic and complex powers. This enables us to state the following definition.

Definition 7. Let $m \in \mathbb{R}$ and $q \in \mathbb{H}$. We call the quaternion $\Lambda(q,m)$ the mth power of q and denote it by q^m.

It is easy to see that the following fact holds.

Proposition 8. *Let $m \in \mathbb{R}$ and $q \in \mathbb{H}$ be such that $|m| \operatorname{Ark}[q] \leqslant \pi$. Then*

$$\operatorname{Ark}\big[q^m\big] = |m| \operatorname{Ark}[q]$$

is satisfied.

2.5. Parameterisation of a ball

Let $n \in \mathbb{N}$. We denote by S^n and B^n the n-dimensional unit sphere and n-dimensional unit closed ball, respectively.

In the sequel we use parameterisation of three-dimensional balls. Let us write

$$I_{\alpha,\beta} = i\cos(\alpha)\cos(\beta) + j\sin(\alpha)\cos(\beta) + k\sin(\beta).$$

It is evident that $I^2_{\alpha,\beta} = -1$ holds. Now we define parameterisation

$$\hat{s} : [-\pi, \pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, \delta] \ni (\alpha, \beta, r) \mapsto rI_{\alpha,\beta}$$

of a ball of radius δ which is contained by the hyperplane $\mathfrak{Re}[q] = 0$.

2.6. Basic notions

Let $T > 0$ denote the period of Eq. (1).

Since the quaternionic multiplication is not commutative there are $m+1$ monomials of order m of the form $q^\mu a_{n,\mu}(t) q^{m-\mu}$ where $a_{m,\mu} \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ is T-periodic. Thus the general form of Eq. (1) we investigate in the sequel is (2), where $n \geqslant 3$. It is worth to mention that we allow terms of order 2 which are not investigated in [1] and [5], i.e. $a_{2,0}(t)q^2$ and $q^2 a_{2,2}(t)$. Of course, we do not allow all possible monomials e.g. $b(t)q^\mu a_{n,\mu}(t)q^{m-\mu}c(t)$ for $b, c \in \mathcal{C}(\mathbb{R}, \mathbb{H})$. The reason is that, such additional coefficients make technical aspects of calculations much more difficult (see Remark 15).

By the change of variables

$$p = q^{-1},$$

the term $q^\mu a_{m,\mu}(t) q^{m-\mu}$ becomes

$$-p^{1-\mu} a_{m,\mu}(t) p^{1+\mu-n},$$

so Eq. (2) is well defined and the vector field is continuous at the point ∞ only if $m \in \{0,1\}$ or $(m,\mu) = (2,1)$. Thus the Poincaré map $\varphi_{(\sigma,T)}$ is much more complicated than the Möbius transformation (cf. [5]).

In the sequel we write $\operatorname{Ark}(0) = 0$. Let $a_{m,\mu} \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ be the coefficients of the vector field v from Eq. (2). We define $\tau_m \geqslant 0$ and $\hat{\tau}_m \geqslant 0$ to be the smallest numbers such that $\operatorname{Ark}[a_{m,\mu}(t)] \leqslant \tau_m$ and $\operatorname{Ark}[-a_{m,\mu}(t)] \leqslant \hat{\tau}_m$ hold for every $t \in \mathbb{R}$ and $\mu \in \{0, 1, \ldots, m\}$.

Let $-\infty \leqslant \alpha < \omega \leqslant \infty$ and $s : (\alpha, \omega) \to \mathbb{H}$ be the full solution of (1). We call s *forward blowing up* (shortly *f.b.*) or *backward blowing up* (*b.b.*) if $\omega < \infty$ or $\alpha > -\infty$, respectively. If $-\infty < \alpha < \omega < \infty$ then s is called *backward and forward blowing up* (*b.f.b.*).

2.7. Fixed points

In the present subsection we state the following improvement of the Brouwer fixed point theorem.

Lemma 9. *Let $m \geqslant 1$ and X be a nonempty convex and compact subset of $\mathbb{R}^m$. Let $f \in \mathcal{C}(X, X)$ be such that there exists $\epsilon > 0$ such that for every $x, y \in X$, $0 < |x - y| < \epsilon$ the inequality*

$$\left| f(x) - f(y) \right| < |x - y| \tag{16}$$

holds. Then there exists exactly one fixed point $x_0 \in X$ of f. Moreover, x_0 is asymptotically stable and attracting in X.

We use it instead of the quaternionic version of the Wolff–Denjoy theorem.

3. Existence

In the present section we prove the existence of periodic solutions inside some sectors. The method we use gives no information on the uniqueness of these solutions inside the sectors.

3.1. Nontrivial free term

We investigate the case $a_{0,0} \not\equiv 0$. This allows us to control the solution starting from the origin. The following theorem corresponds to [2, Theorem 4].

Theorem 10. *Let $n \geqslant 3$, $a_{1,0}, a_{1,1} \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $a_{l,\mu} \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ for $l \in \{0, 2, 3, \dots, n\}$, $\mu \in \{0, 1, \dots, l\}$ be T-periodic. If there exists number $M \geqslant n$ such that*

$$a_{0,0} \not\equiv 0 \quad \textit{and} \quad \hat{\tau}_0 < \frac{\pi}{M-1}, \tag{17}$$

$$\begin{cases} \tau_l < \dfrac{l-1}{M-1}\pi, & \textit{for } 2 \leqslant l \leqslant \dfrac{M+1}{2}, \\ \tau_l \leqslant \dfrac{M-l}{M-1}\pi, & \textit{for } \dfrac{M+1}{2} < l \leqslant n, \end{cases} \tag{18}$$

$$\sum_{l=2}^{n} \sum_{\mu=0}^{l} |a_{l,\mu}| \not\equiv 0 \tag{19}$$

hold, then Eq. (2) *has in the sector $\mathcal{S}(\frac{\pi}{M-1})$*

- *at least one T-periodic solution ξ,*
- *infinitely many forward blowing up solutions.*

Moreover, the equation

$$\dot{q} = v_1(t, q) = \sum_{l=0}^{n} (-1)^l \sum_{\mu=0}^{l} q^{\mu} a_{l,\mu}(t) q^{l-\mu} \tag{20}$$

has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$

- *at least one T-periodic solution χ,*
- *infinitely many backward blowing up solutions.*

If in addition we assume that the equalities

$$a_{l,\mu} \equiv 0, \quad \textit{for all odd numbers } l \geqslant 3 \tag{21}$$

hold, then Eq. (2) *has*

- *at least one T-periodic solution χ in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$,*
- *infinitely many backward blowing up solutions in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$.*

Proof. Our goal is to define a compact set $E \subset \mathbb{H}$, such that $\operatorname{int} E$ is homeomorphic to four-dimensional unit ball and the inclusion

$$\varphi^{-1}_{(0,T)}(E) \subset \operatorname{int} E \tag{22}$$

holds. It allows us to apply the Brouwer fixed point theorem and get the existence of at least one periodic solution inside the set E.

We set

$$0 < \varepsilon < \min\left\{\frac{\pi}{M-1} - \hat{\tau}_0, \frac{\frac{l-1}{M-1}\pi - \tau_l}{l-1}, \text{ for } 2 \leqslant l \leqslant n\right\}$$

and

$$A(\varepsilon) = \left\{q \in \mathbb{H}\colon \operatorname{Ark}[q] \leqslant \frac{\pi}{M-1} - \varepsilon\right\}.$$

Let us recall that $0 \in A(\varepsilon)$.

We show that the vector field v points outward or is tangent to the set $A(\varepsilon)$ in every point from $\partial A(\varepsilon) \setminus \{0\}$.

Let $q \in \partial A(\varepsilon) \setminus \{0\}$, i.e. $\operatorname{Ark}[q] = \frac{\pi}{M-1} - \varepsilon$. An outward orthogonal vector $n(q)$ to $A(\varepsilon)$ at $q = \mathfrak{Re}[q] + I_q|\mathfrak{Im}[q]|$ is equal to $I_q q = -|\mathfrak{Im}[q]| + I_q \mathfrak{Re}[q]$. Now we calculate the inner product of $n(q)$ and $v(t,q)$. Let $2 \leqslant l \leqslant \frac{M+1}{2}$ and $\mu \in \{0, 1, \dots, l\}$. Since

$$-\frac{\pi}{2} + (l-1)\frac{\pi}{M-1} - (l-1)\varepsilon = \frac{\pi}{2}\frac{2l-1-M}{M-1} - (l-1)\varepsilon \in \left(-\frac{\pi}{2} + \tau_l, 0\right)$$

and, by (13),

$$\begin{aligned}\operatorname{Ark}\left[\overline{I_q} q^{l-1}\right] &= \left|\operatorname{Arg}\left[\bar{i}\left(\mathfrak{Re}[q] + i\left|\mathfrak{Im}[q]\right|\right)^{l-1}\right]\right| = \left|\operatorname{Arg}\left[e^{i(-\frac{\pi}{2} + (l-1)\frac{\pi}{M-1} - (l-1)\varepsilon)}\right]\right| \\ &< \frac{\pi}{2} - \tau_l,\end{aligned}$$

then, by (9), Remark 6 and Proposition 3, the estimation

$$\begin{aligned}\left\langle n(q), q^{\mu} a_{l,\mu}(t) q^{l-\mu}\right\rangle &= \mathfrak{Re}\left[\overline{(I_q q)} q^{\mu} a_{l,\mu}(t) q^{l-\mu}\right] = \mathfrak{Re}\left[q^{l-\mu}\,\overline{(I_q q)} q^{\mu} a_{l,\mu}(t)\right] \\ &= \mathfrak{Re}\left[\overline{I_q}\bar{q} q^l a_{l,\mu}(t)\right] = |q|^2\,\mathfrak{Re}\left[\overline{I_q} q^{l-1} a_{l,\mu}(t)\right] \\ &= |q|^{l+1}\left|a_{l,\mu}(t)\right|\cos\left(\mathrm{Ark}\left[\overline{I_q} q^{l-1} a_{l,\mu}(t)\right]\right) \\ &\geqslant 0 \end{aligned} \tag{23}$$

holds for every $t \in \mathbb{R}$. Now let $M \geqslant n \geqslant l > \frac{M+1}{2}$ and $\mu \in \{0, 1, \ldots, l\}$. Since

$$\begin{aligned} -\frac{\pi}{2} + (l-1)\frac{\pi}{M-1} - (l-1)\varepsilon &= \frac{\pi}{2}\,\frac{2l-1-M}{M-1} - (l-1)\varepsilon \\ &\in \left(-\frac{\pi}{2} + \tau_l,\ \frac{\pi}{2}\,\frac{2l-1-M}{M-1}\right) \\ &\subset \left(-\frac{\pi}{2} + \tau_l,\ \frac{\pi}{2}\right) \end{aligned}$$

and

$$\mathrm{Ark}\left[\overline{I_q} q^{l-1}\right] < \max\left\{\frac{\pi}{2} - \tau_l,\ \frac{\pi}{2}\,\frac{2l-1-M}{M-1}\right\},$$

then, as previously, the estimation

$$\begin{aligned}\left\langle n(q), q^{\mu} a_{l,\mu}(t) q^{l-\mu}\right\rangle &= |q|^{l+1}\left|a_{l,\mu}(t)\right|\cos\left(\mathrm{Ark}\left[\overline{I_q} q^{l-1} a_{l,\mu}(t)\right]\right) \\ &\geqslant 0\end{aligned}$$

holds. Let now $l = 1$. Then

$$\begin{aligned}\left\langle n(q), q^{\mu} a_{l,\mu}(t) q^{l-\mu}\right\rangle &= \mathfrak{Re}\left[\overline{I_q}\bar{q} q^l a_{l,\mu}(t)\right] = |q|^2\,\mathfrak{Re}\left[\overline{I_q} a_{l,\mu}(t)\right] \\ &= 0.\end{aligned} \tag{24}$$

Finally, in the case $l = 0$ we get

$$\begin{aligned}\left\langle n(q), a_{0,0}(t)\right\rangle &= \mathfrak{Re}\left[\overline{I_q q} a_{0,0}(t)\right] = \mathfrak{Re}\left[(-\overline{I_q q})\left(-a_{0,0}(t)\right)\right] \\ &\geqslant 0,\end{aligned}$$

because of (13) and

$$\mathrm{Ark}\left[(-\overline{I_q q})\right] = \left|\frac{\pi}{2} + \frac{\pi}{M-1} - \varepsilon - \pi\right| = \frac{\pi}{2}\,\frac{M-3}{M-1} + \varepsilon < \frac{\pi}{2} - \hat{\tau}_0.$$

By (17), $v(t,0)$ points outward $A(\varepsilon)$ or $v(t,0) = 0$.

Summing up, every monomial of v points (in every point of $\partial A(\varepsilon)$) outwards $A(\varepsilon)$ or is tangent to $\partial A(\varepsilon)$. Since $a_{0,0} \not\equiv 0$, every solution starting from $\partial A(\varepsilon)$ leaves $A(\varepsilon)$ in time shorter than T.

We fix $\varepsilon_0 > 0$ such that the set $A(\varepsilon_0)$ has properties described above. We set $A = A(\varepsilon_0)$.

Let $m = \frac{M-1}{2} \geqslant 1$ and $\gamma > 0$. We define three-dimensional surface Θ_γ in $\mathbb{H}$ to be

$$\Theta_\gamma = s\left([-\pi,\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[0, \tan\left(\frac{\pi}{2} - m\varepsilon_0\right)\right]\right)$$

where

$$s(\alpha,\beta,o) = \gamma[1 + oI_{\alpha,\beta}]^{-\frac{1}{m}},$$

cf. Section 2.5. Thus Θ_γ is homeomorphic to the closed three-dimensional unit ball. Moreover, the conditions

$$\Theta_\gamma \subset A,$$
$$\mathrm{Ark}\left[s\left([-\pi,\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left\{\tan\left(\frac{\pi}{2} - m\varepsilon_0\right)\right\}\right)\right] = \frac{\pi}{M-1} - \varepsilon_0,$$
$$s\left([-\pi,\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[0, \tan\left(\frac{\pi}{2} - m\varepsilon_0\right)\right)\right) \subset \operatorname{int} A$$

hold. Thus $A \setminus \Theta_\gamma$ consists of two connected components: bounded and unbounded one. We define the set E_γ to be the union of the bounded one and the set Θ_γ.

Now we investigate behaviour of the vector field v on the set Θ_γ. First of all, let us notice that an outward orthogonal vector $n(\alpha,\beta,o)$ to E_γ at the point $s(\alpha,\beta,o)$ has the form

$$n(\alpha,\beta,o) = (1 - oI_{\alpha,\beta})^{\frac{m+1}{m}}.$$

Now we check the inner product of the outward orthogonal vector and every of the monomials of the vector field v. For $l = 0$ we get

$$\begin{aligned}\left\langle n(\alpha,\beta,o), a_{0,0}(t)\right\rangle &= \mathfrak{Re}\left[(1 + oI_{\alpha,\beta})^{\frac{m+1}{m}} a_{0,0}(t)\right] \\ &\geqslant -|1 + oI_{\alpha,\beta}|^{\frac{m+1}{m}} N_0 \\ &\geqslant -N_0 \sin^{-\frac{m+1}{m}}(m\varepsilon_0),\end{aligned} \tag{25}$$

where $N_0 = \max\{|a_{0,0}|\colon t \in \mathbb{R}\}$ and

$$1 \leqslant |1 + oI_{\alpha,\beta}| \leqslant \frac{1}{\sin(m\varepsilon_0)}.$$

For $l = 1$ and $\mu \in \{0,1\}$ we get

$$\begin{aligned}\left\langle n(\alpha,\beta,o), s^\mu(\alpha,\beta,o) a_{1,\mu}(t) s^{1-\mu}(\alpha,\beta,o)\right\rangle &= \mathfrak{Re}\left[\gamma(1 + oI_{\alpha,\beta}) a_{1,\mu}(t)\right] \\ &\geqslant -\gamma|1 + oI_{\alpha,\beta}| N_1 \\ &\geqslant -\gamma N_1 \sin^{-1}(m\varepsilon_0),\end{aligned} \tag{26}$$

where $N_1 = \max\{|a_{1,1}| + |a_{1,0}|\colon t \in \mathbb{R}\}$. Now for $l \geqslant 2$ and $\mu \in \{0,1,\ldots,l\}$ we get

$$\mathrm{Ark}\left[(1 + oI_{\alpha,\beta})^{\frac{m+1-l}{m}}\right] \geqslant \left(\frac{\pi}{2} - m\varepsilon_0\right)\left|\frac{m+1-l}{m}\right|,$$

so when $l-1 \leqslant m$ (i.e. $l \leqslant \frac{M+1}{2}$)

$$\begin{aligned}\operatorname{Ark}\bigl[(1+oI_{\alpha,\beta})^{\frac{m+1-l}{m}}a_{l,\mu}(t)\bigr] &< \left(\frac{\pi}{2}-m\varepsilon_0\right)\frac{m+1-l}{m}+\frac{l-1}{2m}\pi-(l-1)\varepsilon_0 \\ &= \frac{\pi}{2}-m\varepsilon_0\end{aligned}$$

and when $l-1>m$

$$\begin{aligned}\operatorname{Ark}\bigl[(1+oI_{\alpha,\beta})^{\frac{m+1-l}{m}}a_{l,\mu}(t)\bigr] &\leqslant \left(\frac{\pi}{2}-m\varepsilon_0\right)\frac{l-m-1}{m}+\frac{2m+1-l}{2m}\pi \\ &= \frac{\pi}{2}-(l-m-1)\varepsilon_0,\end{aligned}$$

thus there exists $\widetilde{m}>0$ such that

$$\operatorname{Ark}\bigl[(1+oI_{\alpha,\beta})^{\frac{m+1-l}{m}}a_{l,\mu}(t)\bigr] < \frac{\pi}{2}-\widetilde{m}$$

holds. Finally,

$$\begin{aligned}\bigl\langle n(\alpha,\beta,o), s^{\mu}(\alpha,\beta,o)a_{l,\mu}(t)s^{l-\mu}(\alpha,\beta,o)\bigr\rangle &= \mathfrak{Re}\bigl[\gamma^l(1+oI_{\alpha,\beta})^{\frac{m+1-l}{m}}a_{l,\mu}(t)\bigr] \\ &\geqslant \gamma^l|1+oI_{\alpha,\beta}|^{\frac{m+1-l}{m}}\bigl|a_{l,\mu}(t)\bigr|\sin(\widetilde{m}) \\ &\geqslant \gamma^l\bigl|a_{l,\mu}(t)\bigr|\sin(\widetilde{m})\min\bigl\{1,\sin^{\frac{l-m-1}{m}}(m\varepsilon_0)\bigr\} \qquad (27)\end{aligned}$$

holds.

If the inequality

$$\sum_{l=2}^{n}\sum_{\mu=0}^{l}\bigl|a_{l,\mu}(t)\bigr|>0 \qquad (28)$$

holds for every $t\in\mathbb{R}$ instead of (19), then, by (25), (26) and (27), we get

$$\begin{aligned}\bigl\langle n(\alpha,\beta,o), v\bigl(t,s^{\mu}(\alpha,\beta,o)\bigr)\bigr\rangle &\geqslant -|1+oI_{\alpha,\beta}|^{\frac{m+1}{m}}N_0-\gamma|1+oI_{\alpha,\beta}|N_1 \\ &\quad+\sin(\widetilde{m})\gamma^2\sum_{l=2}^{n}\sum_{\mu=0}^{l}|1+oI_{\alpha,\beta}|^{\frac{m+1-l}{m}}\bigl|a_{l,\mu}(t)\bigr| \\ &> 0\end{aligned}$$

provided that γ is big enough. In this case the vector field v points outward E_γ in every point of Θ_γ. We choose such a γ_0 and write $E=E_{\gamma_0}$. Then (22) holds.

When (28) is not satisfied but (19) holds, then the vector field v may point inward E_γ in some point of Θ_γ and some times t. But let us notice that the family of sets Θ_γ is the foliation of the sector A, i.e.

$$A=\{0\}\cup\bigcup_{\gamma\in\mathbb{R}}\Theta_\gamma$$

holds. Let $t_0 \in [0,T]$, $\nu > 0$ be such that $t_0+\nu < T$ and $|a_{l,\mu}(t)| > \kappa > 0$ for some $l \geqslant 2$, $\mu \in \{0,1,\dots,l\}$ and every $t \in [t_0, t_0+\nu]$. Then there exists γ_0 such that for every fixed $\gamma \geqslant \gamma_0$ and every $q \in \Theta_\gamma$ if $\varphi_{(0,t_0)}(q) \in E_\gamma$ then $\varphi_{(t_0,\nu)}(\varphi_{(0,t_0)}(q)) \notin E_\gamma$. It is due to the fact, that, by (25) and (26), the trajectory can move inward E_γ in the exponential speed in t, because the vector field is bounded by linear term in γ. Later, by (27), the trajectory is swept out the set E_γ in time shorter than ν, because the vector field has at least quadratic growth in γ, i.e. at least $\gamma^2\kappa\sin(\widetilde{m})\min\{1, \sin^{\frac{l-m-1}{m}}(m\varepsilon_0)\}$.

We write $E = E_{\gamma_0}$. Then (22) holds.

This proves that there exists at least one periodic solution inside $\mathcal{S}(\frac{\pi}{M-1})$.

We now prove the existence of infinitely many f.b. solutions inside the sector $\mathcal{S}(\frac{\pi}{M-1})$, i.e. we prove the existence of solutions $\eta : (-\infty, \omega_\eta) \to \mathbb{H}$ of Eq. (2) inside the sector satisfying $\omega_\eta < \infty$ and

$$\lim_{t \to \omega_\eta^-} \eta(t) = \infty. \tag{29}$$

Let $t_0 \in [0,T]$, $\nu > 0$ be such that $t_0 + 2\nu < T$ and $|a_{l,\mu}(t)| > \kappa > 0$ for some $l \geqslant 2$, $\mu \in \{0,1,\dots,l\}$ and every $t \in [t_0, t_0+2\nu]$. We define the set

$$D(\delta) = \bigcup_{\gamma \geqslant \delta} \Theta_\gamma \tag{30}$$

for some $\delta > 0$. We choose such a big δ that for every $\gamma \geqslant \delta$ the vector field $-v$ points outward $D(\gamma)$ in every point of Θ_γ (i.e. v points inward $D(\gamma)$ in every point of $\Theta_\gamma \setminus \partial A$ and points outward $D(\gamma)$ in every point of $\Theta_\gamma \cap \partial A$). Moreover, it is possible to choose such a big δ, that every full solution η of (2) satisfying $\eta(t) \in D(\delta)$ for $t \in [t_0, t_0+\nu]$ leaves $D(\delta)$ in time shorter than ν or is not defined for time $t+\nu$. It is possible due to the fact that, by (27), on every set Θ_γ the component of the dominating term of v which is normal to Θ_γ has growth at least quadratic in γ, i.e. it is at least $\gamma^2\kappa\sin(\widetilde{m})\min\{1, \sin^{\frac{l-m-1}{m}}(m\varepsilon_0)\}$.

Let ϕ be the local flow on $\mathbb{R}\times\mathbb{H}$ generated by (2). Write

$$K = [t_0, t_0+2\nu] \times D(\delta),$$
$$\widetilde{D}(\delta) = D(\delta) \cap \left\{ q \in \mathbb{H}\colon \operatorname{Ark}[q] = \frac{\pi}{M-1} - \varepsilon_0 \right\}.$$

Here

$$K^- = [t_0, t_0+2\nu] \times \widetilde{D}(\delta) \cup \{t_0+2\nu\} \times D(\delta),$$
$$K^+ = [t_0, t_0+2\nu] \times \Theta_\delta \cup \{t_0\} \times D(\gamma).$$

Thus K and K^- are closed sets in $\mathbb{R}\times\mathbb{H}$, so, by Proposition 1, K is a Ważewski set for the local flow ϕ.

We fix $\theta \in (t_0, t_0+\nu)$. To obtain a contradiction, we suppose that $\{\theta\}\times\Theta_\delta \subset K^*$. By Lemma 2, the map $\Delta : \{\theta\}\times\Theta_\delta \ni (\theta,p) \mapsto \phi(\sigma(\theta,p),(\theta,p)) \in K^-$ is continuous (here σ is the escape-time function of K). But by the above arguments $\sigma(\theta,p) < \nu$, so, since $\pi_1(\phi(\sigma(\theta,p),(\theta,p))) = \theta + \sigma(\theta,p)$ holds (here $\pi_1 : \mathbb{R}\times\mathbb{H} \to \mathbb{R}$ is the projection on the time variable), we get $\Delta(\{\theta\}\times\Theta_\delta) \subset [t_0, t_0+2\nu) \times \widetilde{D}(\delta)$. But $\sigma(\theta,p) = 0$ for every $p \in \Theta_\delta \cap \widetilde{D}(\delta)$, so $\Delta|_{\{\theta\}\times[\Theta_\delta\cap\widetilde{D}(\delta)]} = \mathrm{id}\,|_{\{\theta\}\times[\Theta_\delta\cap\widetilde{D}(\delta)]}$. Moreover, $[t_0, t_0+2\nu) \times \widetilde{D}(\delta)$ is homotopy equivalent to $\{\theta\}\times[\Theta_\delta \cap \widetilde{D}(\delta)]$. Finally, since $\{\theta\}\times[\Theta_\delta\cap\widetilde{D}(\delta)]$ is homeomorphic to S^2 and $\{\theta\}\times\Theta_\delta$ is homeomorphic to B^3, we get a retraction of B^3 to S^2 which is the desired contradiction.

Summing up, we have proved that there exists $p \in \Theta_\delta$ such that $\phi^+(\theta,p) \subset K$, i.e. for the solution η of (2) satisfying $\eta(\theta) = p$ one gets $\omega_\eta < \infty$. But, by (30), for every $\theta < t < \omega_\eta$ there exists $\gamma > \delta$

such that $\eta(t) \in \Theta_\gamma$ and γ increases when t increases, so (29) holds. Since no solution can enter the set A, $\eta(t) \in A$ for every $t < \theta$, so $\alpha_\eta = -\infty$. By the arbitrariness of the choice of θ we get infinitely many b.f. solutions inside A which finishes the proof of the main part of the theorem.

By the equality

$$v(t, -q) = v_1(t, q), \tag{31}$$

the Poincaré map $\varphi_{(0,T)} : (-E) \to \operatorname{int}(-E)$ of (20) is well defined. Thus there exists at least one T-periodic solution of Eq. (20) inside the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$. Moreover, there are infinitely many b.b. solutions inside the sector.

Let now $a_{1,1} = a_{1,0} \equiv 0$ hold. Then the assumption (21) implies the equalities $v(t,q) = v_1(t,q)$ and $v(t,-q) = v_1(t,q)$ hold for every $(t,z) \in \mathbb{R} \times \mathbb{H}$. Thus the properties of the sets E and $-E$ described as above are preserved. If we allow $a_{1,1} \not\equiv 0$ or $a_{1,0} \not\equiv 0$ then the behaviour of the vector field v on the boundaries of the sets is qualitatively the same: the terms $qa_{1,1}$ and $a_{1,0}q$ are tangent or dominated by $\sum_{l=2}^{n} \sum_{\mu=0}^{l} q^\mu a_{l,\mu}(t) q^{l-\mu}$. Finally, Eq. (2) has in every sector $\mathcal{S}(\frac{\pi}{M-1})$ and $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$ at least one T-periodic solution and infinitely many f.b. and b.b., respectively, ones. $\square$

Remark 11. It is possible to allow $a_{1,0}, a_{1,1} \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ in Theorem 10 provided that the equality

$$\mathfrak{Im}[a_{1,0}] = -\mathfrak{Im}[a_{1,1}] \tag{32}$$

holds. Indeed, we get in the crucial (24)

$$\begin{aligned} \langle n(q), qa_{1,1}(t) + a_{1,0}(t)q \rangle &= \mathfrak{Re}\big[\overline{I_q}\bar{q}q\big[a_{1,1}(t) + a_{1,0}(t)\big]\big] \\ &= |q|^2\, \mathfrak{Re}\big[\overline{I_q}\big[a_{1,1}(t) + a_{1,0}(t)\big]\big] \\ &= |q|^2\big[a_{1,1}(t) + a_{1,0}(t)\big]\, \mathfrak{Re}[\overline{I_q}] \\ &= 0, \end{aligned}$$

which changes nothing on the boundary of $A(\varepsilon)$. It does not affect the estimation (26) either.

Example 12. By Theorem 10, the equation

$$\dot{q} = q^3 - 1 - \epsilon + j + e^{it}$$

has at least one 2π-periodic solution inside $\mathcal{S}(\frac{\pi}{2})$ for every $\epsilon > 0$. Here $M = 3$, $\tau_3 = \tau_2 = 0$ and $\hat{\tau}_0 = \arctan(\epsilon^{-1}) < \frac{\pi}{2}$.

Example 13. By Theorem 10, the equation

$$\dot{q} = -q^3 + q(1+k)q + \sin(t)q - \frac{3}{2} + ie^{it} + e^{jt}$$

has at least one 2π-periodic solution inside $\widehat{\mathcal{S}}(\frac{\pi}{2})$. Here $M = 3$, $\tau_3 = 0$, $\tau_2 = \frac{\pi}{4} < \frac{\pi}{2}$ and $\hat{\tau}_0 = \arctan(\frac{2}{3-2\sqrt{2}}) < \frac{\pi}{2}$.

Example 14. By Theorem 10 and Remark 11, the equation

$$\dot{q} = q^4 + iq - qi - 1 + j\sin(t) + k$$

has at least one 2π-periodic solution inside of every sector $\mathcal{S}(\frac{\pi}{3})$ and $\widehat{\mathcal{S}}(\frac{\pi}{3})$. Here $M=4$, $\tau_4=\tau_3=\tau_2=0$ and $\hat{\tau}_0=\arctan(\sqrt{2})<\frac{\pi}{3}$.

Remark 15. The equality (23) is the place which does not allow (2) to contain monomials of the form $a(t)q^{\mu}b(t)q^{l-\mu}c(t)$. Indeed, even if we control $\mathrm{Ark}[a(t)b(t)c(t)]$ and get in (23)

$$\begin{aligned}\langle n(q), a(t)q^{\mu}b(t)q^{l-\mu}c(t)\rangle &= \mathfrak{Re}\big[\overline{(I_q q)}a(t)q^{\mu}b(t)q^{l-\mu}c(t)\big]\\ &= \frac{1}{|a(t)c(t)|^2}\,\mathfrak{Re}\big[\bar{c}(t)q^{l-\mu}c(t)\overline{I_q}\bar{q}a(t)q^{\mu}\bar{a}(t)\{a(t)b(t)c(t)\}\big],\end{aligned}$$

then despite the equalities

$$\mathrm{Ark}\big[\bar{a}(t)q^{\mu}a(t)\big]=\mathrm{Ark}\big[q^{\mu}\big],$$
$$\mathrm{Ark}\big[\bar{c}(t)q^{l-\mu}c(t)\big]=\mathrm{Ark}\big[q^{l-\mu}\big]$$

the inequality

$$\mathrm{Ark}\big[\bar{c}(t)q^{l-\mu}c(t)\overline{I_q}\bar{q}a(t)q^{\mu}\bar{a}(t)\big]>\mathrm{Ark}\big[q^{l-\mu}\overline{I_q}\bar{q}q^{\mu}\big]$$

may occur, e.g. for $q=1+i$, $l=\mu=2$, $c\equiv 1$, and $a(t)=2+j$ one gets $I_q=i$ and

$$\mathrm{Ark}\big[-i(1-i)(2+j)(1+i)^2(2-j)\big]=\mathrm{Ark}[3-3i-4j+4k]=\arctan\left(\frac{\sqrt{41}}{3}\right)>\frac{\pi}{4},$$
$$\mathrm{Ark}\big[-i(1-i)(1+i)^2\big]=\mathrm{Ark}[1-i]=\frac{\pi}{4}.$$

3.2. Trivial free term

In this section we investigate the equation

$$\dot{q}=v(t,q)=\sum_{l=1}^{n}\sum_{\mu=0}^{l}q^{\mu}a_{l,\mu}(t)q^{l-\mu}. \tag{33}$$

Linear term is the dominating one in the neighbourhood of the origin. If it satisfies the inequality

$$\int_0^T a_{1,1}(t)+a_{1,0}(t)\,dt<0, \tag{34}$$

then it is possible to adapt method from the proof of Theorem 10. In this case $qa_{1,1}(t)+a_{1,0}(t)q$ plays similar role to $a_{0,0}(t)$.

We state the main theorem of the section.

Theorem 16. *Let $n\geqslant 3$, $a_{1,1}(t), a_{1,0}(t)\in\mathcal{C}(\mathbb{R},\mathbb{R})$ and $a_{l,\mu}\in\mathcal{C}(\mathbb{R},\mathbb{H})$ for $l\in\{2,3,\dots,n\}$, $\mu\in\{0,1,\dots,l\}$ be T-periodic. If there exists number $M\geqslant n$ such that* (34), (18) *and* (19) *are satisfied, then Eq.* (33) *has in the sector $\mathcal{S}(\frac{\pi}{M-1})$*

- *at least one T-periodic solution,*
- *infinitely many f.b. solutions.*

Moreover, the equation

$$\dot{q} = v_1(t,q) = \sum_{l=1}^{n} (-1)^l \sum_{\mu=0}^{l} q^{\mu} a_{l,\mu}(t) q^{l-\mu} \tag{35}$$

has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$

- *at least one* T*-periodic solution,*
- *infinitely many b.b. solutions.*

Proof. We adopt notation and modify the proof of Theorem 10. Let $0 < \delta < 1$. We define $E(\delta) = E \cap \{q \in \mathbb{H}\colon\ |q| \geqslant \delta\}$ and $\widehat{E}(\delta) = E \cap \{q \in \mathbb{C}\colon\ |q| = \delta\}$.

Since $a_{1,1}$ and $a_{1,0}$ take only real values, the solution of the equation

$$\dot{q} = a_{1,1}(t)q + q a_{1,0}(t) \tag{36}$$

is given by

$$\eta(t) = e^{\int_0^t a_{1,1}(\tau) + a_{1,0}(\tau)\,d\tau} \eta(0), \tag{37}$$

so

$$\left|\eta(T)\right| < \left|\eta(0)\right| \tag{38}$$

holds for $\eta(0) \neq 0$. But the term $a_{1,1}(t)q + q a_{1,0}(t)$ is the dominating one in the neighbourhood of the origin. Thus (38) holds also for Eq. (33), provided that $|\eta(0)|$ is small enough.

Thus $\varphi_{(0,-T)}(E(\delta)) \subset E(\delta)$ is satisfied, which finishes the proof of the main part of the theorem.

As previously, the case of Eq. (35) comes from the equality (31). $\square$

Remark 17. As in Section 3.1, it is possible to allow $a_{1,1}(t), a_{1,0}(t) \in \mathcal{C}(\mathbb{R}, \mathbb{H})$, provided that (32) is satisfied. Since the solution of (36) may not be given by (37) (see Example 18) we need to use different approach.

Write $f(t) = |q(t)|^2$ where q is a fixed solution of (36). Then

$$\begin{aligned}
\frac{d}{dt} f(t) &= \frac{d}{dt}\bigl\langle q(t), q(t)\bigr\rangle = 2\bigl\langle \dot{q}(t), q(t)\bigr\rangle = 2\bigl\langle q(t), a_{1,1}(t)q(t) + q(t)a_{1,0}(t)\bigr\rangle \\
&= 2\,\mathfrak{Re}\bigl[\bar{q}(t)\bigl(a_{1,1}(t)q(t) + q(t)a_{1,0}(t)\bigr)\bigr] = 2\bigl|q(t)\bigr|^2\,\mathfrak{Re}\bigl[a_{1,1}(t) + a_{1,0}(t)\bigr] \\
&= 2f(t)\,\mathfrak{Re}\bigl[a_{1,1}(t) + a_{1,0}(t)\bigr],
\end{aligned}$$

so, by (34), we get $|f(T)| < |f(0)|$ if only $f(0) \neq 0$. Finally, we get the crucial (38), thus Remark 11 finishes the proof of the statement.

Example 18. Let $f \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ be 2-periodic and given by

$$f(t) = \begin{cases} i + 2tj, & \text{for } t \in [0,1], \\ i + 2(2-t)j, & \text{for } t \in [1,2]. \end{cases}$$

Then the solution of the linear equation

$$\dot{q} = f(t)q$$

is not given by

$$\eta(t) = e^{\int_0^t f(\tau)\,d\tau}\eta(0).$$

To see this let us differentiate the function η in the interval $(0,1)$. Having in mind (11), we get

$$\begin{aligned}
\frac{d}{dt}\eta(t) &= \frac{d}{dt}\big[e^{t(i+tj)}\eta(0)\big] = \left[\frac{d}{dt}e^{t(i+tj)}\right]\eta(0)\\
&= \left[\sum_{m=0}^{\infty}\frac{1}{m!}\frac{d}{dt}\big[t(i+tj)\big]^m\right]\eta(0) = \left[\sum_{m=0}^{\infty}\frac{1}{m!}\frac{d}{dt}\big(t^m(i+tj)^m\big)\right]\eta(0)\\
&= \left[\sum_{m=0}^{\infty}\frac{1}{m!}\left[\left(\frac{d}{dt}t^m\right)(i+tj)^m + t^m\left(\frac{d}{dt}(i+tj)^m\right)\right]\right]\eta(0)\\
&= \left[\sum_{m=1}^{\infty}\frac{t^{m-1}(i+tj)^m}{(m-1)!}\right]\eta(0) + \left[\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{d}{dt}(i+tj)^m\right]\eta(0)\\
&= (i+tj)e^{t(i+tj)}\eta(0) + \left[\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{d}{dt}(i+tj)^m\right]\eta(0),
\end{aligned}$$

so it is enough to show that $\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{d}{dt}(i+tj)^m \neq tje^{t(i+tj)}$. Let us observe that

$$\frac{2ti + j(t^2-1)}{1+t^2}(i+tj) = (i+tj)j,$$

$$j(i+tj) = (i+tj)\frac{2ti + j(t^2-1)}{1+t^2}$$

hold. So, since $\frac{2ti+j(t^2-1)}{1+t^2} = j + 2\frac{ti-j}{1+t^2}$, we get

$$\begin{aligned}
\frac{d}{dt}(i+tj)^m &= \sum_{n=0}^{m-1}(i+tj)^n j (i+tj)^{m-1-n}\\
&= \sum_{n\ \mathrm{odd}}\frac{2ti + j(t^2-1)}{1+t^2}(i+tj)^n(i+tj)^{m-1-n} + \sum_{n\ \mathrm{even}} j(i+tj)^n(i+tj)^{m-1-n}\\
&= \left(jm + 2\left\lfloor\frac{m}{2}\right\rfloor\frac{ti-j}{1+t^2}\right)(i+tj)^{m-1},
\end{aligned}$$

where $\lfloor x\rfloor$ is the integer part of x, i.e.

$$\lfloor x\rfloor = \max\{n\in\mathbb{Z}\colon\ n\leqslant x\}.$$

Thus

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{d}{dt} (i+tj)^m = tje^{t(i+tj)} + \sum_{m=0}^{\infty} \frac{t^m}{m!} 2 \left\lfloor \frac{m}{2} \right\rfloor \frac{ti-j}{1+t^2} (i+tj)^{m-1} \tag{39}$$

holds, so it is enough to show that the second component of the right-hand side of (39) is nonzero. To do that we notice that

$$\frac{ti-j}{1+t^2}(i+tj) = k,$$
$$(i+tj)^2 = -(t^2+1)$$

hold and calculate

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{t^m}{m!} 2 \left\lfloor \frac{m}{2} \right\rfloor \frac{ti-j}{1+t^2} (i+tj)^{m-1} \\
&= \sum_{l=0}^{\infty} \frac{t^{2l}}{(2l)!} 2 \left\lfloor \frac{2l}{2} \right\rfloor \frac{ti-j}{1+t^2} (i+tj)^{2l-1} + \sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} 2 \left\lfloor \frac{2l+1}{2} \right\rfloor \frac{ti-j}{1+t^2} (i+tj)^{2l} \\
&= \sum_{l=1}^{\infty} \frac{t^{2l}}{(2l)!} 2lk(-1)^{l-1} \left(1+t^2\right)^{l-1} + \sum_{l=1}^{\infty} \frac{t^{2l+1}}{(2l+1)!} 2l \frac{ti-j}{1+t^2} (-1)^l \left(1+t^2\right)^l \\
&= kt^2 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{(2l-1)!} \left[t^2\left(1+t^2\right)\right]^{l-1} + (ti-j)t^3 \sum_{l=1}^{\infty} \frac{(-1)^l}{(2l+1)!} 2l \left[t^2\left(1+t^2\right)\right]^{l-1} \\
&= kt^2 X + (ti-j)t^3 Y \\
&\neq 0
\end{aligned}$$

because $X > 0$ and $Y < 0$. To see that let us observe that if we write $X = \sum_{l=1}^{\infty} x_l$, then for every odd l we get $x_l + x_{l+1} > 0$ for every $t \in [0,1]$. In the case of Y we get just $y_l + y_{l+1} < 0$.

Example 19. By Theorem 16, the equation

$$\dot{q} = q^4\left(1 + k\sqrt{3}\cos(t)\right) + \left(2\sin(t) - 1\right)q$$

has at least one 2π-periodic solution inside $\mathcal{S}(\frac{2\pi}{9})$. Here $M = \frac{11}{2}$, $\tau_4 = \frac{\pi}{3} \leqslant \frac{\pi}{3}$ and $\tau_3 = \tau_2 = 0$.

Now we investigate the situation when (34) is not satisfied. Unfortunately we are unable to prove the following theorem assuming only

$$\int_0^T a_{1,1}(t) + a_{1,0}(t)\, dt \geqslant 0. \tag{40}$$

We need to assume that

$$a_{1,1}(t) + a_{1,0}(t) \geqslant 0 \quad \text{for every } t \in \mathbb{R} \tag{41}$$

holds.

Theorem 20. *Let $n \geqslant 3$, $a_{1,1}(t), a_{1,0}(t) \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $a_{l,\mu} \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ for $l \in \{2, 3, \ldots, n\}$, $\mu \in \{0, 1, \ldots, l\}$ be T-periodic. If there exists number $M \geqslant n$ such that* (41), (18) *and* (19) *are satisfied, then Eq.* (33) *has in the sector $\mathcal{S}(\frac{\pi}{M-1})$*

- *no periodic solutions,*
- *infinitely many f.b. solutions,*
- *the trivial solution is repelling in the whole sector.*

Moreover, Eq. (35) *has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$*

- *no periodic solutions,*
- *infinitely many b.b. solutions,*
- *the trivial solution is attracting in the whole sector.*

If, in addition, the condition

$$a_{l,\mu} \equiv 0 \quad \textit{for all even numbers } l \geqslant 2 \tag{42}$$

is satisfied, then Eq. (33) *has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$*

- *no periodic solutions,*
- *infinitely many f.b. solutions,*
- *the trivial solution is repelling in the whole sector.*

Proof. We use the notation from the proof of Theorem 10. Since

$$A = \{0\} \cup \bigcup_{\gamma > 0} \Theta_\gamma,$$

it is enough to show that every component $q^\mu a_{l,\mu}(t) q^{l-\mu}$ of the vector field points outward the set E_γ in every point of $[\partial E_\gamma] \setminus \{0\}$. But almost all was done in the proof of Theorem 10. It remains to show this for $q^\mu a_{1,\mu}(t) q^{1-\mu}$ and points from Θ_γ, i.e. to make a different estimation in (26). By (41), we write

$$\begin{aligned}
\big\langle n(\alpha, \beta, o), s(\alpha, \beta, o) a_{1,1}(t) + a_{1,0}(t) s(\alpha, \beta, o)\big\rangle &= \mathfrak{Re}\big[\gamma(1 + o I_{\alpha,\beta})\big[a_{1,1}(t) + a_{1,0}(t)\big]\big] \\
&= \gamma\big[a_{1,1}(t) + a_{1,0}(t)\big]\,\mathfrak{Re}[1 + o I_{\alpha,\beta}] \\
&= \gamma\big[a_{1,1}(t) + a_{1,0}(t)\big] \\
&\geqslant 0
\end{aligned}$$

which finishes the proof of the main part of theorem.

The case of Eq. (35) comes from (31).

If (42) is satisfied, then $v(t, -q) = -v(t, q)$ holds for every $t \in \mathbb{R}$ and $q \in \mathbb{H}$, so every component of the vector field v points outward the set $-E_\gamma$ in every point of $-[\partial E_\gamma] \setminus \{0\}$. □

Remark 21. It is possible to allow $a_{1,0}, a_{1,1} \in c(\mathbb{R}, \mathbb{H})$ in Theorem 20 provided that (32) is satisfied. Indeed it does not change the crucial estimations for linear terms.

Open problem 22. Is it possible to prove Theorem 20 assuming (40) instead of (41)?

Example 23. By Theorem 20, the equation

$$\dot{q} = q^3 + q\big(1 + j\sin(t)\big)q + q$$

has no periodic solution inside $\mathcal{S}(\frac{\pi}{2})$. Here $M = 3$, $\tau_3 = 0$ and $\tau_2 = \frac{\pi}{4} < \frac{\pi}{2}$.

Example 24. By Theorem 20, the equation

$$\dot{q} = q^3\big[1 + i\sin(t)\big]q + (2 + j)q^2 + q$$

has no periodic solution inside $\mathcal{S}(\frac{\pi}{4})$. By Theorem 16, the equation has at least one 2π-periodic solution inside $\widehat{\mathcal{S}}(\frac{\pi}{4})$. Here $M = 5$, $\tau_4 = \frac{\pi}{4} \leqslant \frac{\pi}{4}$ and $\tau_2 = \arctan(\frac{1}{2}) < \frac{\pi}{4}$.

Example 25. By Theorem 20, the equation

$$\dot{q} = q^3 + \sin(t)kq - \sin(t)qk$$

has no periodic solution inside $\mathcal{S}(\frac{\pi}{2})$ and $\widehat{\mathcal{S}}(\frac{\pi}{2})$. Here $M = 3$, $\tau_3 = \tau_2 = 0$.

4. Uniqueness

In the present section we prove uniqueness of periodic solutions inside some sectors. We use the strengthened version of the Brouwer fixed point theorem given in Lemma 9. To do that we need to impose more restrictive assumptions than in Section 3. In fact, we are able to provide results only in the case of nontrivial free term described in Section 3.1.

Theorem 26. *Let $n \geqslant 3$, $a_{1,0}, a_{1,1} \in \mathcal{C}(\mathbb{R}, [0,\infty))$ and $a_{l,\mu} \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ for $l \in \{0, 2, 3, \dots, n\}$, $\mu \in \{0, 1, \dots, l\}$ be T-periodic. If there exists number $M \geqslant 2n - 1$ such that* (17), (19) *and*

$$\begin{cases} \tau_l < \dfrac{l-1}{M-1}\pi, & \text{for } 2 \leqslant l \leqslant \dfrac{M+3}{4}, \\ \tau_l \leqslant \dfrac{M-2l+1}{2(M-1)}\pi, & \text{for } \dfrac{M+3}{4} < l \leqslant n \end{cases} \tag{43}$$

hold, then Eq. (2) *has in the sector $\mathcal{S}(\frac{\pi}{M-1})$*

- *exactly one T-periodic solution ξ. It is asymptotically unstable and repelling in the whole sector,*
- *infinitely many forward blowing up solutions.*

Moreover, Eq. (20) *has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$*

- *exactly one T-periodic solution χ. It is asymptotically stable and attracting in the whole sector,*
- *infinitely many backward blowing up solutions.*

If, in addition, we assume that (21) *holds, then Eq.* (2) *has*

- *exactly one T-periodic solution χ in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$. It is asymptotically stable and attracting in the whole sector,*
- *infinitely many backward blowing up solutions in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$,*
- *infinitely many solutions which are heteroclinic from ξ to χ.*

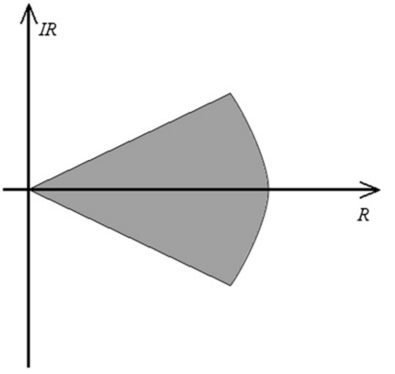

Fig. 1. The set $E_\gamma \cap \mathbb{R}[I]$ is marked in grey. For every I such that $I^2 = -1$ the sets $E_\gamma \cap \mathbb{R}[I]$ look the same.

Proof. First of all, let us notice that, since $M \geqslant 2n - 1$, (43) implies (18). Thus, by Theorem 10, there exist in the sector $\mathcal{S}(\frac{\pi}{M-1})$ at least one T-periodic solution ξ and infinitely many forward blowing up solutions. So it is enough to prove that ξ is asymptotically unstable and repelling in the whole sector. We do it using Lemma 9.

Let E be as in the proof of Theorem 10. By (17), there exists $\rho_0 > 0$ such that for every $q \in E \cap \{q \in \mathbb{H}\colon\ \mathfrak{Re}[q] \leqslant \rho_0\}$ one gets $\varphi_{(0,T)}(q) \notin E$. So for every $0 < \rho \leqslant \rho_0$ and $\widehat{E}(\rho) = E \cap \{q \in \mathbb{H}\colon\ \mathfrak{Re}[q] \geqslant \rho\}$ the inclusion $\varphi_{(0,-T)}(\widehat{E}(\rho)) \subset \widehat{E}(\rho)$ is satisfied. Moreover, by analysis of construction of E (E is a convex subset of the complex plane rotated along the real axis, see Fig. 1), it is easy to see that $\widehat{E}(\rho)$ is a convex and compact subset of H. Let us fix ρ. Then there exists $\hat{\gamma}$ such that for every $q \in \widehat{E}(\rho)$ we get

$$\rho \leqslant |q| < \hat{\gamma}. \tag{44}$$

Now it remains to verify the last assumption of Lemma 9, i.e. (16). Let us write

$$\rho > \epsilon > 0. \tag{45}$$

Then, by the continuous dependence of solutions on the initial conditions and compactness of $\widehat{E}(\rho)$, there exists $\hat{\epsilon}$, $\epsilon \geqslant \hat{\epsilon} > 0$ such that for every $p, q \in \widehat{E}(\rho)$ satisfying $0 < |p - q| < \hat{\epsilon}$ the estimation

$$\left|\varphi_{(0,-t)}(p), \varphi_{(0,-t)}(q)\right| < \epsilon \tag{46}$$

holds for every $t \in [0, T]$. Let us fix η, ζ, $\eta \neq \zeta$, solutions of (2) such that $\eta(0), \zeta(0) \in \widehat{E}(\rho)$ and (46) are satisfied, i.e.

$$\left|\eta(-t) - \zeta(-t)\right| < \epsilon$$

for every $t \in [0, T]$. Then to get

$$\left|\eta(-T) - \zeta(-T)\right| < \left|\eta(0) - \zeta(0)\right|$$

it is enough the function

$$f(t) = \left|\eta(-t) - \zeta(-t)\right|^2$$

be decreasing, i.e.

$$\begin{aligned}\frac{d}{dt}f(t) &= 2\left\langle \eta(-t)-\zeta(-t), \frac{d}{dt}\eta(-t)-\frac{d}{dt}\zeta(-t)\right\rangle \\ &= 2\big\langle \eta(-t)-\zeta(-t), -v\big(-t,\eta(-t)\big)+v\big(-t,\zeta(-t)\big)\big\rangle \\ &\leqslant 0\end{aligned}$$

holds for every $t\in[-T,0]$ and for some t the inequality is strict. To sum up, we need to show that

$$\big\langle \eta-\zeta, v(-t,\eta)-v(-t,\zeta)\big\rangle \geqslant 0 \tag{47}$$

holds for every $t\in\mathbb{R}$ and for some t the inequality is strict. We do that by analysing the contribution of every monomial of v to the above inner product. We start with $l=0$

$$\begin{aligned}\langle \eta-\zeta, a_{0,0}-a_{0,0}\rangle &= \mathfrak{Re}\big[\overline{(\eta-\zeta)}(a_{0,0}-a_{0,0})\big] \\ &= 0.\end{aligned}$$

Let now $l=1$ and $\mu\in\{0,1\}$

$$\begin{aligned}\big\langle \eta-\zeta, \eta^{\mu}a_{l,\mu}\eta^{l-\mu}-\zeta^{\mu}a_{l,\mu}\zeta^{l-\mu}\big\rangle &= \mathfrak{Re}\big[\overline{(\eta-\zeta)}\big(\eta^{\mu}a_{l,\mu}\eta^{l-\mu}-\zeta^{\mu}a_{l,\mu}\zeta^{l-\mu}\big)\big] \\ &= \mathfrak{Re}\big[\overline{(\eta-\zeta)}(\eta-\zeta)a_{1,\mu}\big] \\ &= |\eta-\zeta|^2\,\mathfrak{Re}[a_{1,\mu}] \\ &\geqslant 0.\end{aligned} \tag{48}$$

In the case $l\geqslant 2$ and $\mu\in\{0,1,\dots,l\}$ we get

$$\begin{aligned}&\big\langle \eta-\zeta, \eta^{\mu}a_{l,\mu}\eta^{l-\mu}-\zeta^{\mu}a_{l,\mu}\zeta^{l-\mu}\big\rangle \\ &\quad= \mathfrak{Re}\big[\overline{(\eta-\zeta)}\big(\eta^{\mu}a_{l,\mu}\eta^{l-\mu}-\zeta^{\mu}a_{l,\mu}\eta^{l-\mu}+\zeta^{\mu}a_{l,\mu}\eta^{l-\mu}-\zeta^{\mu}a_{l,\mu}\zeta^{l-\mu}\big)\big] \\ &\quad= \mathfrak{Re}\big[\overline{(\eta-\zeta)}\big(\eta^{\mu}-\zeta^{\mu}\big)a_{l,\mu}\eta^{l-\mu}\big]+\mathfrak{Re}\big[\overline{(\eta-\zeta)}\zeta^{\mu}a_{l,\mu}\big(\eta^{l-\mu}-\zeta^{l-\mu}\big)\big] \\ &\quad= \mathfrak{Re}\big[\eta^{l-\mu}\overline{(\eta-\zeta)}\big(\eta^{\mu}-\zeta^{\mu}\big)a_{l,\mu}\big]+\mathfrak{Re}\big[\big(\eta^{l-\mu}-\zeta^{l-\mu}\big)\overline{(\eta-\zeta)}\zeta^{\mu}a_{l,\mu}\big].\end{aligned} \tag{49}$$

Let $\eta(-t)=\zeta(-t)+\nu(-t)$ where $|\nu(-t)|<\epsilon$. Keeping in mind (45) and the fact that for $\mu\geqslant 2$ one gets

$$(\zeta+\nu)^{\mu} = \zeta^{\mu}+\nu\zeta^{\mu-1}+\zeta\nu\zeta^{\mu-2}+\dots+\zeta^{\mu-1}\nu+P_{\mu}(\nu,\zeta),$$

where P_{μ} is a polynomial such that

$$\big|P_{\mu}(\nu,\zeta)\big| \leqslant 2^{\mu}|\nu|^2|\zeta|^{\mu-2},$$

we deal with the first component of (49) and obtain

$$\begin{aligned}&\mathfrak{Re}\big[\eta^{l-\mu}\overline{(\eta-\zeta)}\big(\eta^{\mu}-\zeta^{\mu}\big)a_{l,\mu}\big] \\ &\quad= \mathfrak{Re}\big[\eta^{l-\mu}\overline{\nu}\big((\zeta+\nu)^{\mu}-\zeta^{\mu}\big)a_{l,\mu}\big] \\ &\quad= \mathfrak{Re}\big[\eta^{l-\mu}\overline{\nu}\big(\nu\zeta^{\mu-1}+\zeta\nu\zeta^{\mu-2}+\dots+\zeta^{\mu-1}\nu\big)a_{l,\mu}\big]+\mathfrak{Re}\big[\eta^{l-\mu}\overline{\nu}P_{\mu}(\nu,\zeta)a_{l,\mu}\big].\end{aligned}$$

Now, by Propositions 3 and 5, we obtain

$$\begin{aligned}
\operatorname{Ark}\bigl[\bar\nu\zeta^{\alpha}\nu\zeta^{\mu-1-\alpha}\bigr] &\leqslant \operatorname{Ark}\bigl[\bar\nu\zeta^{\alpha}\nu\bigr]+\operatorname{Ark}\bigl[\zeta^{\mu-1-\alpha}\bigr]\\
&= \operatorname{Ark}\bigl[\zeta^{\alpha}\bigr]+(\mu-1-\alpha)\operatorname{Ark}[\zeta]\\
&= (\mu-1)\operatorname{Ark}[\zeta]\\
&\leqslant (\mu-1)\frac{\pi}{M-1}-(\mu-1)\varepsilon_0
\end{aligned}$$

for every $\alpha\in\{0,1,\dots,\mu\}$, where ε_0 is as in the proof of Theorem 10. So, by the fact that (43) implies

$$\tau_l\leqslant\frac{M-2l+1}{2(M-1)}\pi,\quad \text{for every } 2\leqslant l\leqslant n,$$

we get

$$\begin{aligned}
&\operatorname{Ark}\bigl[\eta^{l-\mu}\bar\nu\bigl(\nu\zeta^{\mu-1}+\zeta\nu\zeta^{\mu-2}+\dots+\zeta^{\mu-1}\nu\bigr)a_{l,\mu}\bigr]\\
&\quad\leqslant (l-\mu)\frac{\pi}{M-1}-(l-\mu)\varepsilon_0+(\mu-1)\frac{\pi}{M-1}-(\mu-1)\varepsilon_0+\frac{M-2l+1}{2(M-1)}\pi\\
&\quad=\frac{\pi}{2}-(l-1)\varepsilon_0
\end{aligned}$$

and

$$\mathfrak{Re}\bigl[\eta^{l-\mu}\bar\nu\bigl(\nu\zeta^{\mu-1}+\zeta\nu\zeta^{\mu-2}+\dots+\zeta^{\mu-1}\nu\bigr)a_{l,\mu}\bigr]\geqslant\mu|\nu|^2|\eta|^{l-\mu}|\zeta|^{\mu-1}|a_{l,\mu}|\sin\bigl((l-1)\varepsilon_0\bigr).$$

Thus, by (44), we obtain the estimation

$$\begin{aligned}
&\mathfrak{Re}\bigl[\eta^{l-\mu}\overline{(\eta-\zeta)}\bigl(\eta^{\mu}-\zeta^{\mu}\bigr)a_{l,\mu}\bigr]\\
&\quad\geqslant\mu|\nu|^2|\eta|^{l-\mu}|\zeta|^{\mu-1}|a_{l,\mu}|\sin\bigl((l-1)\varepsilon_0\bigr)-2^{\mu}|\eta|^{l-\mu}|\nu|^3|\zeta|^{\mu-2}|a_{l,\mu}|\\
&\quad=\mu|\eta|^{l-\mu}|\nu|^2|\zeta|^{\mu-2}|a_{l,\mu}|\left[|\zeta|\sin\bigl((l-1)\varepsilon_0\bigr)-\frac{2^{\mu}}{\mu}|\nu|\right]\\
&\quad=\mu|\eta|^{l-\mu}|\nu|^2|\zeta|^{\mu-2}|a_{l,\mu}|\bigl[\rho\sin(\varepsilon_0)-2^{n}\epsilon\bigr]\\
&\quad\geqslant 0
\end{aligned}\tag{50}$$

provided that

$$0<\epsilon<2^{-n}\rho\sin(\varepsilon_0)\tag{51}$$

holds.

For $\mu=0$ we get

$$\mathfrak{Re}\bigl[\eta^{l-\mu}\overline{(\eta-\zeta)}\bigl(\eta^{\mu}-\zeta^{\mu}\bigr)a_{l,\mu}\bigr]=0$$

and for $\mu=1$

$$\begin{aligned}\mathfrak{Re}\big[\eta^{l-\mu}\overline{(\eta-\zeta)}\big(\eta^{\mu}-\zeta^{\mu}\big)a_{l,\mu}\big] &= |\nu|^2\,\mathfrak{Re}\big[\eta^{l-1}a_{l,\mu}\big]\\ &\geqslant |\nu|^2|\eta|^{l-1}|a_{l,\mu}|\sin\big[(l-1)\varepsilon_0\big]\\ &\geqslant 0. \end{aligned}\tag{52}$$

Summing up, the inequality

$$\mathfrak{Re}\big[\eta^{l-\mu}\overline{(\eta-\zeta)}\big(\eta^{\mu}-\zeta^{\mu}\big)a_{l,\mu}\big]\geqslant 0$$

holds provided that (51) is satisfied. Similarly, it is possible to show that

$$\mathfrak{Re}\big[\big(\eta^{l-\mu}-\zeta^{l-\mu}\big)\overline{(\eta-\zeta)}\zeta^{\mu}a_{l,\mu}\big]\geqslant 0$$

holds.

It is worth to mention that there are strict inequalities in (50) and (52) when $|a_{l,\mu}(t)|>0$, so, by (19), we get

$$\big\langle \eta-\zeta,\ \eta^{\mu}a_{l,\mu}\eta^{l-\mu}-\zeta^{\mu}a_{l,\mu}\zeta^{l-\mu}\big\rangle>0$$

for some $l\geqslant 2$, $\mu\in\{0,1,\dots,l\}$ and $t\in[0,T]$ (and the weak inequality for all l, μ and t) which finishes the proof of the main part of theorem.

As previously, the case of Eq. (20) comes from the equality (31).

Let (21) hold and η be a solution of (2) such that $\eta(t_0)=0$ for some $t_0\in\mathbb{R}$. Then, by (17), $\eta(t)\in\mathcal{S}(\frac{\pi}{M-1})$ for every $t<t_0$ so $\lim_{t\to-\infty}|\eta(t)-\xi(t)|=0$. Moreover, $\eta(t)\in\widehat{\mathcal{S}}(\frac{\pi}{M-1})$ for every $t>t_0$ so $\lim_{t\to\infty}|\eta(t)-\chi(t)|=0$, i.e. η is heteroclinic from ξ to χ. $\square$

Remark 27. As in Theorem 10, it is possible to allow $a_{1,0},a_{1,1}\in\mathcal{C}(\mathbb{R},\mathbb{H})$ in Theorem 26 provided that the equality (32) holds. Indeed, we get

$$\begin{aligned}\big\langle \eta-\zeta,\ \eta^{\mu}a_{l,\mu}\eta^{l-\mu}-\zeta^{\mu}a_{l,\mu}\zeta^{l-\mu}\big\rangle &= \mathfrak{Re}\big[\overline{(\eta-\zeta)}(\eta a_{1,1}+a_{1,0}\eta-\zeta a_{1,1}-a_{1,0}\zeta)\big]\\ &= \mathfrak{Re}\big[\overline{(\eta-\zeta)}(\eta-\zeta)(a_{1,1}+a_{1,0})\big]\\ &= |\eta-\zeta|^2\,\mathfrak{Re}[a_{1,1}+a_{1,0}]\\ &\geqslant 0\end{aligned}$$

instead of (48) which has the same contribution to the (47).

Open problem 28. Is it possible to prove Theorem 26 with condition (18) instead of (43) and $M\geqslant n$ instead of $M\geqslant 2n-1$, i.e. for wider sectors?

Example 29. By Theorem 26, the equation

$$\dot{q}=q^3-1-\epsilon-je^{it}$$

has exactly one 2π-periodic solution inside $\mathcal{S}(\frac{\pi}{4})$ for every $\epsilon>0$. Here $M=5$, $\tau_3=\tau_2=0$ and $\hat{\tau}_0=\arctan(\frac{1}{1+\epsilon})<\frac{\pi}{4}$.

Example 30. By Theorem 26, the equation

$$\dot{q} = q^3 + q(1+\epsilon+k)q + \left[1+\sin(t)\right]q - \frac{3}{2} + e^{jt}$$

has exactly one 2π-periodic solution inside $\widehat{\mathcal{S}}(\frac{\pi}{4})$ for every $\epsilon > 0$. Here $M = 5$, $\tau_3 = 0$, $\tau_2 = \arctan(\frac{1}{1+\epsilon}) < \frac{\pi}{4}$ and $\hat{\tau}_0 = \arcsin(\frac{2}{3}) < \frac{\pi}{4}$.

Example 31. By Theorem 26 and Remark 27, the equation

$$\dot{q} = q^4 + iq - qi - 1 + \frac{k}{2}e^{jt}$$

has exactly one 2π-periodic solution inside $\mathcal{S}(\frac{\pi}{6})$. It is asymptotically unstable and repelling in the whole $\mathcal{S}(\frac{\pi}{6})$. Moreover, the equation has exactly one 2π-periodic solution inside $\widehat{\mathcal{S}}(\frac{\pi}{6})$. It is asymptotically stable and attracting in the whole $\widehat{\mathcal{S}}(\frac{\pi}{6})$. Here $M = 7$, $\tau_3 = \tau_2 = 0$ and $\hat{\tau}_0 = \arctan(\frac{1}{2}) < \frac{\pi}{6}$.

Acknowledgment

The author was supported by Polish Ministry of Science and Higher Education grant No. N N201 549038.

References

[1] P. Wilczyński, Quaternionic-valued ordinary differential equations. The Riccati equation, J. Differential Equations 247 (7) (2009) 2163–2187, doi:10.1016/j.jde.2009.06.015.

[2] P. Wilczyński, Planar nonautonomous polynomial equations. II. Coinciding sectors, J. Differential Equations 246 (7) (2009) 2762–2787, doi:10.1016/j.jde.2008.11.009.

[3] J. Campos, J. Mawhin, Periodic solutions of quaternionic-valued ordinary differential equations, Ann. Mat. Pura Appl. (4) 185 (Suppl.) (2006) S109–S127.

[4] J.D. Gibbon, D.D. Holm, R.M. Kerr, I. Roulstone, Quaternions and particle dynamics in the Euler fluid equations, Nonlinearity 19 (8) (2006) 1969–1983.

[5] H. Żołądek, Classification of diffeomorphisms of $\mathbb{S}^4$ induced by quaternionic Riccati equations with periodic coefficients, Topol. Methods Nonlinear Anal. 33 (2) (2009) 205–215.

[6] A. Gasull, J. Llibre, X. Zhang, One-dimensional quaternion homogeneous polynomial differential equations, J. Math. Phys. 50 (8) (2009) 082705, doi:10.1063/1.3139115, 17 pp.

[7] X. Zhang, Global structure of quaternion polynomial differential equations, Comm. Math. Phys. 303 (2) (2011) 301–316, doi:10.1007/s00220-011-1196-y.

[8] P. Wilczyński, Planar nonautonomous polynomial equations: The Riccati equation, J. Differential Equations 244 (6) (2008) 1304–1328, doi:10.1016/j.jde.2007.12.008.

[9] R. Fueter, Über einen Hartogs'schen Satz, Comment. Math. Helv. 12 (1939) 75–80.

[10] H. Leutwiler, Modified quaternionic analysis in $\mathbb{R}^3$, Compl. Variabl. Theory Appl. 20 (1–4) (1992) 19–51.

[11] C.G. Cullen, An integral theorem for analytic intrinsic functions on quaternions, Duke Math. J. 32 (1965) 139–148.

[12] R. Srzednicki, Ważewski method and Conley index, in: Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2004, pp. 591–684.

[13] I.R. Porteous, Topological Geometry, Van Nostrand/Reinhold Co., London, 1969.